



Generalized Fractional Hermite-Hadamard Inequalities for Twice Differentiable s -Convex Functions

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Abstract. Some Hermite-Hadamard type inequalities via Riemann-Liouville fractional integral for twice differentiable functions having some s -convexity of second kind properties are established. A class of s -affine of second kind functions is identified such as these inequalities are sharp.

1. Introduction and Preliminary Results

Let \mathbb{R} be the set of real numbers, $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$ be a convex function in the classical sense, i.e. a function which satisfies the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1)$$

whenever $x, y \in I$ and $t \in [0, 1]$. The functions that make the inequality (1) sharp are called affine functions. Let $a, b \in I$ with $a < b$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \quad (2)$$

named after C. Hermite and J. Hadamard [7]. Inequality (2) can be considered as a necessary and sufficient condition for a function to be convex. It is sharp for all affine functions. Due to its great utility in mathematics development and in applications, this inequality was extended to various classes of generalized convex functions, and refined under additional hypotheses (see [3–6, 10–15]). The sharpness of the new inequalities is not taken into account sometimes.

More authors recently established Hermite-Hadamard type inequalities based on Riemann-Liouville fractional integrals (see [8]) for different kinds of convex functions (see for example [10, 14, 15]), many times

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without taking into account their sharpness. In this paper, we consider the class of s -convex functions of second kind, the Breckner convexity (see [1, 2]). We intend to derive several new fractional Hermite-Hadamard type inequalities for classes of Breckner's s -convex functions. Also, we intend to study their sharpness, identifying a class of functions, as large as possible, that make sharp the new inequalities. Improving these inequalities would mean, in this condition, enlarging their sharpness class.

Let us suppose that $s \in [0, 1]$.

Definition 1.1 ([1]). A function $f : I \rightarrow (0, \infty)$ is said to be an s -convex function in the second sense (also called Breckner convex), if

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y), \quad \forall x, y \in I, t \in [0, 1]. \tag{3}$$

The functions, which make the inequality (3) sharp are called s -affine in the second sense. Note that for $s = 1$ the Breckner convexity reduces to the concept of classical convexity.

Definition 1.2 ([8]). Let $f \in L_1[a, b]$. Then the Riemann-Liouville integrals $J_{a^+}^\alpha f$ and $J_{b^-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) dt, \quad x > a, \tag{4}$$

and

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) dt, \quad x < b, \tag{5}$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt,$$

is the well known Gamma function.

In this section, we need to prove the following preliminary result, which plays a key role in the next development.

Lemma 1.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable function on (a, b) with $a < b$. If $f'' \in L[a, b]$ and $n \in \mathbb{N}^*$, then, following equality for fractional integrals holds

$$\mathcal{G}(n, \alpha, a, b)(f) = \frac{(b - a)^2}{(n + 1)^2} \int_0^1 (1 - t)^{\alpha+1} \left[f'' \left(\frac{n + t}{n + 1} a + \frac{1 - t}{n + 1} b \right) + f'' \left(\frac{1 - t}{n + 1} a + \frac{n + t}{n + 1} b \right) \right] dt, \tag{6}$$

where

$$\begin{aligned} \mathcal{G}(n, \alpha, a, b)(f) &= \frac{(n + 1)^\alpha \Gamma(\alpha + 2)}{(b - a)^\alpha} \left[J_{\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)^-}^\alpha f(a) + J_{\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right)^+}^\alpha f(b) \right] \\ &\quad - \frac{b - a}{n + 1} \left[f' \left(\frac{1}{n + 1} a + \frac{n}{n + 1} b \right) - f' \left(\frac{n}{n + 1} a + \frac{1}{n + 1} b \right) \right] \\ &\quad - (\alpha + 1) \left[f \left(\frac{1}{n + 1} a + \frac{n}{n + 1} b \right) + f \left(\frac{n}{n + 1} a + \frac{1}{n + 1} b \right) \right]. \end{aligned}$$

Proof. Let us compute

$$\begin{aligned} & \int_0^1 (1-t)^{\alpha+1} \left[f'' \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right) + f'' \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right) \right] dt \\ &= \int_0^1 (1-t)^{\alpha+1} f'' \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right) dt + \int_0^1 (1-t)^{\alpha+1} f'' \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right) dt \\ &= I_1 + I_2. \end{aligned} \tag{7}$$

Integrate by parts two times successively, in order to compute each integral in (6). In case of I_1 one obtains:

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^{\alpha+1} f'' \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right) dt \\ &= \frac{n+1}{b-a} f' \left(\frac{n}{n+1}a + \frac{1}{n+1}b \right) - \frac{(\alpha+1)(n+1)^2}{(b-a)^2} f \left(\frac{n}{n+1}a + \frac{1}{n+1}b \right) \\ &\quad + \frac{\alpha(\alpha+1)(n+1)^2}{(b-a)^2} \int_0^1 (1-t)^{\alpha-1} f \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right) dt \\ &= \frac{n+1}{b-a} f' \left(\frac{n}{n+1}a + \frac{1}{n+1}b \right) - \frac{(\alpha+1)(n+1)^2}{(b-a)^2} f \left(\frac{n}{n+1}a + \frac{1}{n+1}b \right) \\ &\quad + \frac{(n+1)^{\alpha+2}}{(b-a)^{\alpha+2}} \Gamma(\alpha+2) J_{\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)^+}^\alpha f(a). \end{aligned} \tag{8}$$

Also, the second integral I_2 becomes:

$$\begin{aligned} I_2 &= \int_0^1 (1-t)^{\alpha+1} f'' \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right) dt \\ &= -\frac{n+1}{b-a} f' \left(\frac{1}{n+1}a + \frac{n}{n+1}b \right) - \frac{(\alpha+1)(n+1)^2}{(b-a)^2} f \left(\frac{1}{n+1}a + \frac{n}{n+1}b \right) \\ &\quad + \frac{\alpha(\alpha+1)(n+1)^2}{(b-a)^2} \int_0^1 (1-t)^{\alpha-1} f \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right) dt \\ &= -\frac{n+1}{b-a} f' \left(\frac{1}{n+1}a + \frac{n}{n+1}b \right) - \frac{(\alpha+1)(n+1)^2}{(b-a)^2} f \left(\frac{1}{n+1}a + \frac{n}{n+1}b \right) \\ &\quad + \frac{(n+1)^{\alpha+2}}{(b-a)^{\alpha+2}} \Gamma(\alpha+2) J_{\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right)^+}^\alpha f(b). \end{aligned} \tag{9}$$

Now, using (8) and (9) in (7) and after conveniently arranging one obtains the required result. \square

Note that Lemma 1.3 reduces to Lemma 1 in [10] for $n = 1$.

2. Hermite-Hadamard Type Inequalities

In this section we derive fractional Hermite-Hadamard inequalities for s -convex functions of second kind.

Theorem 2.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable function on (a, b) with $a < b$ and $n \in \mathbb{N}^*$. If $f'' \in L[a, b]$ and $|f''|$ is an s -convex function of second kind, then we have the following inequality for fractional integrals:*

$$|\mathcal{G}(n, \alpha, a, b)(f)| \leq \frac{(b-a)^2}{(n+1)^{s+2}} \left[C(s, \alpha, t) + \frac{1}{\alpha + s + 2} \right] [|f''(a)| + |f''(b)|],$$

where

$$C(s, \alpha, t) = \int_0^1 (1-t)^{\alpha+1} (n+t)^s dt. \quad (10)$$

It is worth to mention here that one can calculate the value of $C(s, \alpha, t)$ using some mathematical software (for example Maple).

Proof. Using Lemma 1.3, the fact that $|f''|$ is an s -convex function of second kind and its integrability property according to [2], we have

$$\begin{aligned} |\mathcal{G}(n, \alpha, a, b)(f)| &\leq \left| \frac{(b-a)^2}{(n+1)^2} \int_0^1 (1-t)^{\alpha+1} \left[f'' \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right) \right] dt \right| \\ &\quad + \left| \frac{(b-a)^2}{(n+1)^2} \int_0^1 (1-t)^{\alpha+1} \left[f'' \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right) \right] dt \right| \\ &\leq \frac{(b-a)^2}{(n+1)^2} \int_0^1 (1-t)^{\alpha+1} \left[\left(\frac{n+t}{n+1} \right)^s |f''(a)| + \left(\frac{1-t}{n+1} \right)^s |f''(b)| \right] dt \\ &\quad + \frac{(b-a)^2}{(n+1)^2} \int_0^1 (1-t)^{\alpha+1} \left[\left(\frac{1-t}{n+1} \right)^s |f''(a)| + \left(\frac{n+t}{n+1} \right)^s |f''(b)| \right] dt \\ &= \frac{(b-a)^2}{(n+1)^{s+2}} \left[\int_0^1 (1-t)^{\alpha+1} [(n+t)^s |f''(a)| + (1-t)^s |f''(b)|] dt \right. \\ &\quad \left. + \int_0^1 (1-t)^{\alpha+1} [(1-t)^s |f''(a)| + (n+t)^s |f''(b)|] dt \right] \\ &= \frac{(b-a)^2}{(n+1)^{s+2}} \left[C(s, \alpha, t) + \frac{1}{\alpha+s+2} \right] [|f''(a)| + |f''(b)|]. \end{aligned}$$

This completes the proof. \square

Theorem 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable function on (a, b) with $a < b$ and $n \in \mathbb{N}^*$. If $f'' \in L[a, b]$ and $|f''|^q$ for $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ is an s -convex function of second kind, then, we have the following inequality for fractional integrals:

$$\begin{aligned} |\mathcal{G}(n, \alpha, a, b)(f)| &\leq \frac{(b-a)^2}{(n+1)^{\frac{2q+s}{q}}} \left(\frac{1}{p(\alpha+1)+1} \right)^{\frac{1}{p}} \left(\frac{1}{s+1} \right)^{\frac{1}{q}} \left[\left(\{(1+n)^{s+1} - n^{s+1}\} |f''(a)|^q + |f''(b)|^q \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(|f''(a)|^q + \{(1+n)^{s+1} - n^{s+1}\} |f''(b)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. Using Lemma 1.3, well known Holders's inequality and the fact that $|f''|^q$ is an s -convex function of

second kind, we have

$$\begin{aligned}
 |\mathcal{G}(n, \alpha, a, b)(f)| &\leq \frac{(b-a)^2}{(n+1)^2} \left(\int_0^1 (1-t)^{p(\alpha+1)} \right)^{\frac{1}{p}} \left(\int_0^1 |f'' \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right)|^q dt \right)^{\frac{1}{q}} \\
 &\quad + \frac{(b-a)^2}{(n+1)^2} \left(\int_0^1 (1-t)^{p(\alpha+1)} \right)^{\frac{1}{p}} \left(\int_0^1 |f'' \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right)|^q dt \right)^{\frac{1}{q}} \\
 &= \frac{(b-a)^2}{(n+1)^2} \left(\frac{1}{p(\alpha+1)+1} \right)^{\frac{1}{p}} \left[\left(\int_0^1 |f'' \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right)|^q dt \right)^{\frac{1}{q}} + \left(\int_0^1 |f'' \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right)|^q dt \right)^{\frac{1}{q}} \right] \\
 &\leq \frac{(b-a)^2}{(n+1)^{\frac{2q+s}{q}}} \left(\frac{1}{p(\alpha+1)+1} \right)^{\frac{1}{p}} \left[\int_0^1 ((n+t)^s |f''(a)|^q + (1-t)^s |f''(b)|^q) dt \right]^{\frac{1}{q}} \\
 &\quad + \left[\int_0^1 ((1-t)^s |f''(a)|^q + (n+t)^s |f''(b)|^q) dt \right]^{\frac{1}{q}} \\
 &= \frac{(b-a)^2}{(n+1)^{\frac{2q+s}{q}}} \left(\frac{1}{p(\alpha+1)+1} \right)^{\frac{1}{p}} \left(\frac{1}{s+1} \right)^{\frac{1}{q}} \left[\{(1+n)^{s+1} - n^{s+1}\} |f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}} \\
 &\quad + \left[|f''(a)|^q + \{(1+n)^{s+1} - n^{s+1}\} |f''(b)|^q \right]^{\frac{1}{q}}.
 \end{aligned}$$

This completes the proof. \square

Theorem 2.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable function on (a, b) with $a < b$ and $n \in \mathbb{N}^*$. If $f'' \in L[a, b]$ and $|f''|^q$ for $p, q \geq 1, \frac{1}{p} + \frac{1}{q} = 1$ is an s -convex function of second kind, then, we have the following inequality for fractional integrals:

$$\begin{aligned}
 |\mathcal{G}(n, \alpha, a, b)(f)| &\leq \frac{(b-a)^2}{(n+1)^{\frac{2q+s}{q}}} \left[\left\{ C(q, s, \alpha, t) |f''(a)|^q + \frac{1}{q(\alpha+1)+s+1} |f''(b)|^q \right\}^{\frac{1}{q}} \right. \\
 &\quad \left. + \left\{ \frac{1}{q(\alpha+1)+s+1} |f''(a)|^q + C(q, s, \alpha, t) |f''(b)|^q \right\}^{\frac{1}{q}} \right],
 \end{aligned}$$

where

$$C(q, s, \alpha, t) = \int_0^1 (1-t)^{q(\alpha+1)} (n+t)^s dt. \tag{11}$$

Proof. Using Lemma 1.3, well known Holders’s inequality and the fact that $|f''|^q$ is an s -convex function of

second kind, we have

$$\begin{aligned}
 |\mathcal{G}(n, \alpha, a, b)(f)| &\leq \frac{(b-a)^2}{(n+1)^2} \left(\int_0^1 1 dt \right)^{\frac{1}{p}} \left(\int_0^1 (1-t)^{q(\alpha+1)} \left| f'' \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right) \right|^q dt \right)^{\frac{1}{q}} \\
 &\quad + \frac{(b-a)^2}{(n+1)^2} \left(\int_0^1 1 dt \right)^{\frac{1}{p}} \left(\int_0^1 (1-t)^{q(\alpha+1)} \left| f'' \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right) \right|^q dt \right)^{\frac{1}{q}} \\
 &= \frac{(b-a)^2}{(n+1)^2} \left[\left(\int_0^1 (1-t)^{q(\alpha+1)} \left| f'' \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + \left(\int_0^1 (1-t)^{q(\alpha+1)} \left| f'' \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right) \right|^q dt \right)^{\frac{1}{q}} \right] \\
 &\leq \frac{(b-a)^2}{(n+1)^{\frac{2q+s}{q}}} \left[\left\{ \int_0^1 (1-t)^{q(\alpha+1)} ((n+t)^s |f''(a)|^q + (1-t)^s |f''(b)|^q) dt \right\}^{\frac{1}{q}} \right. \\
 &\quad \left. + \left\{ \int_0^1 (1-t)^{q(\alpha+1)} ((1-t)^s |f''(a)|^q + (n+t)^s |f''(b)|^q) dt \right\}^{\frac{1}{q}} \right] \\
 &= \frac{(b-a)^2}{(n+1)^{\frac{2q+s}{q}}} \left[\left\{ C(q, s, \alpha, t) |f''(a)|^q + \frac{1}{q(\alpha+1) + s + 1} |f''(b)|^q \right\}^{\frac{1}{q}} \right. \\
 &\quad \left. + \left\{ \frac{1}{q(\alpha+1) + s + 1} |f''(a)|^q + C(q, s, \alpha, t) |f''(b)|^q \right\}^{\frac{1}{q}} \right].
 \end{aligned}$$

This completes the proof. \square

Theorem 2.4. Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable function on (a, b) with $a < b$ and $n \in \mathbb{N}^*$. If $f'' \in L[a, b]$ and $|f''|^q$ for $q > 1$ is an s -convex function of second kind, then, we have the following inequality for fractional integrals:

$$\begin{aligned}
 |\mathcal{G}(n, \alpha, a, b)(f)| &\leq \frac{(b-a)^2}{(n+1)^{\frac{2q+s}{q}}} \left(\frac{1}{\alpha+2} \right)^{1-\frac{1}{q}} \left[\left\{ C(s, \alpha, t) |f''(a)|^q + \frac{1}{\alpha+s+2} |f''(b)|^q \right\}^{\frac{1}{q}} \right. \\
 &\quad \left. + \left\{ \frac{1}{\alpha+s+2} |f''(a)|^q + C(s, \alpha, t) |f''(b)|^q \right\}^{\frac{1}{q}} \right],
 \end{aligned}$$

where $C(s, \alpha, t)$ is given by (10).

Proof. Using Lemma 1.3, well known power mean inequality and the fact that $|f''|^q$ is an s -convex function

of second kind, we have

$$\begin{aligned}
 |\mathcal{G}(n, \alpha, a, b)(f)| &\leq \frac{(b-a)^2}{(n+1)^2} \left(\int_0^1 (1-t)^{\alpha+1} \right)^{1-\frac{1}{q}} \left[\int_0^1 (1-t)^{\alpha+1} \left| f'' \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right) \right|^q dt \right]^{\frac{1}{q}} \\
 &\quad + \frac{(b-a)^2}{(n+1)^2} \left(\int_0^1 (1-t)^{\alpha+1} \right)^{1-\frac{1}{q}} \left[\int_0^1 (1-t)^{\alpha+1} \left| f'' \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right) \right|^q dt \right]^{\frac{1}{q}} \\
 &= \frac{(b-a)^2}{(n+1)^2} \left(\frac{1}{\alpha+2} \right)^{1-\frac{1}{q}} \left[\left(\int_0^1 (1-t)^{\alpha+1} \left| f'' \left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + \left(\int_0^1 (1-t)^{\alpha+1} \left| f'' \left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b \right) \right|^q dt \right)^{\frac{1}{q}} \right] \\
 &\leq \frac{(b-a)^2}{(n+1)^{\frac{2q+s}{q}}} \left(\frac{1}{\alpha+2} \right)^{1-\frac{1}{q}} \left\{ \int_0^1 (1-t)^{\alpha+1} ((n+t)^s |f''(a)|^q + (1-t)^s |f''(b)|^q) dt \right\}^{\frac{1}{q}} \\
 &\quad + \left\{ \int_0^1 (1-t)^{\alpha+1} ((1-t)^s |f''(a)|^q + (n+t)^s |f''(b)|^q) dt \right\}^{\frac{1}{q}} \\
 &= \frac{(b-a)^2}{(n+1)^{\frac{2q+s}{q}}} \left(\frac{1}{\alpha+2} \right)^{1-\frac{1}{q}} \left[\left\{ C(s, \alpha, t) |f''(a)|^q + \frac{1}{\alpha+s+2} |f''(b)|^q \right\}^{\frac{1}{q}} \right. \\
 &\quad \left. + \left\{ \frac{1}{\alpha+s+2} |f''(a)|^q + C(s, \alpha, t) |f''(b)|^q \right\}^{\frac{1}{q}} \right].
 \end{aligned}$$

This completes the proof. \square

Remark 2.5. When $n = 1$, one obtains the inequalities proved in [10] for Breckner convex functions.

3. Sharpness

In order to study the sharpness of the inequalities derived in the previous section we focus on the proofs of the above theorems. We remark that the inequalities used for proving the theorems are: the s -convexity of second kind, the subadditivity, and Holder’s inequality. As consequence, we must think about those functions that make sharp these inequalities. As consequence, we identify the sharpness class of each inequality as a subset of the functions having the s -affine of second kind behavior corresponding to the Breckner convexity required by the hypothesis.

The class of functions that are s -affine in the first sense is identified in [9], Corollary 3.3. It is shown that the equality holds in (3) and that the s -affine functions in the second sense are constant, except the case $s = 1$.

Proposition 3.1. *Ifs $\neq 1$ then the inequality in Theorem 2.1 is sharp for every $f \in \{f : I \rightarrow \mathbb{R} | f(x) = px+q, p, q \in \mathbb{R}\}$.*

Proof. First let us suppose that $s \neq 1$. Since $|f''|$ is supposed to be s -convex of second kind and the inequality 3, used in the proof of Theorem 2.1, is supposed to be sharp, then $|f''| = c, c \in \mathbb{R}$. Since f'' is a derivative, it should have Darboux property. As consequence, f'' should be constant, which means that function f should

be at most a second degree polynomial. Let us suppose that $f'' = c$, which implies that $f(x) = \frac{c}{2}x^2 + px + q$, $x \in I$. An elementary calculus gives:

$$f\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right) + f\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) = \frac{c\left[(a^2 + b^2)(n^2 + 1) + 4nab\right]}{2(n+1)^2} + p(a+b) + 2q,$$

$$f'\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right) - f'\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) = \frac{c(n-1)(b-a)}{n+1},$$

and after integrating two times by parts, one gets:

$$J_{\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right)^-}^\alpha f(a) = \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{n+1}\right)^\alpha \left[\frac{c(n^2a^2 + 2nab + b^2)}{(n+1)^2} + p\frac{na+b}{n+1} + q \right]$$

$$- \frac{1}{\Gamma(\alpha+2)} \left(\frac{b-a}{n+1}\right)^{\alpha+1} \left[\frac{c(na+b)}{n+1} + p \right] + \frac{c}{\Gamma(\alpha+3)} \left(\frac{b-a}{n+1}\right)^{\alpha+2};$$

$$J_{\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right)^+}^\alpha f(b) = \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{n+1}\right)^\alpha \left[\frac{c(a^2 + 2nab + n^2b^2)}{(n+1)^2} + p\frac{a+nb}{n+1} + q \right]$$

$$+ \frac{1}{\Gamma(\alpha+2)} \left(\frac{b-a}{n+1}\right)^{\alpha+1} \left[\frac{c(a+nb)}{n+1} + p \right] - \frac{c}{\Gamma(\alpha+3)} \left(\frac{b-a}{n+1}\right)^{\alpha+2}.$$

Introducing these results in the left side of the inequality of Theorem 2.1 and computing one obtains that this part of the inequality equals to zero, whenever $c, p, q \in \mathbb{R}$.

On another hand, it is known that the $C(s, \alpha, t)$ is not expressible by rational functions (eventually generalized, having real exponents) unless the conditions of Tchebycheff fulfill. So, in general, the right side of the inequality in Theorem 2.1 has

$$\frac{(b-a)^2}{(n+1)^{s+2}} \left[C(s, \alpha, t) + \frac{1}{\alpha + s + 2} \right] \neq 0.$$

As consequence, one should have $|f''(a)| + |f''(b)| = 0$ in order to obtain a sharp inequality. This means that $f''(x) = 0$, $x \in I$. As consequence, the inequality is sharp only if $c = 0$, which means $f(x) = px + q$, $x \in I$, with $p, q \in \mathbb{R}$. □

Remark 3.2. If $s = 1$ then the inequality in Theorem 2.1 also becomes sharp for $f \in \{f : I \rightarrow \mathbb{R} | f(x) = px + q, p, q \in \mathbb{R}\}$. Indeed, in this case one has

$$C(1, \alpha, t) = \frac{n}{\alpha + 1} + \frac{1}{(\alpha + 1)(\alpha + 2)}$$

and the right side of the inequality equals to

$$\frac{(b-a)^2}{(n+1)^3} \left[\frac{n}{\alpha + 1} + \frac{1}{(\alpha + 1)(\alpha + 2)} + \frac{1}{\alpha + 3} \right] [|f''(a)| + |f''(b)|].$$

Obviously, this becomes zero only whether $f''(x) \equiv 0$, $x \in I$, which means that f is a first degree polynomial, as required.

Remark 3.3. A similar reasoning proves that if $s \in (0, 1]$ then the inequalities in Theorem 2.2, Theorem 2.3 and Theorem 2.4 are also sharp if $f \in \{f : I \rightarrow \mathbb{R} | f(x) = dx + g, d, g \in \mathbb{R}\}$.

So, in this section we identified a class of sharpness for every inequality derived in this paper. This may not be maximal, from the point of view of inclusion in the set of functions satisfying the requested hypotheses. Getting better inequalities in this context means, according to our results, proving inequalities that are

sharp in a larger class of functions.

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